

Analytic study of holographic superconductors with non-linear electrodynamics

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Abstract

In this paper, we have analytically studied the properties of the s-wave holographic superconductors in the planar Schwarzschild-AdS background. Using the Sturm-Liouville eigenvalue problem we have been able to calculate the critical temperature for condensation in terms of the charge density. This is a continuation and eventual culmination of the work begun in [26]. The results obtained analytically agree well with the numerical findings[22].

1 Introduction

For the past few years, the AdS/CFT duality[1]-[2], which provides an exact correspondence between a gravity theory in (d+1)-dimensional AdS space to that with a strongly coupled gauge theory living on d-dimensions has been extensively applied in order to describe various phenomena in usual condensed matter physics including high T_c superconductivity. The holographic description of s-wave superconductors basically consists of a charged planar AdS black hole minimally coupled to a complex scalar field. The formation of scalar hair below the critical temperature (T_c) triggers the superconductivity in the boundary field theory through the mechanism of spontaneous $U(1)$ symmetry breaking[3]-[7].

Till date, several investigations have been performed in various directions in order to understand a number of crucial properties of these holographic superconductors[8]-[26]. Most of these models consider the usual framework of Maxwell electrodynamics[8]-[21], whereas, on the other hand, some of the investigations have also been performed in the framework of non-linear electrodynamics[22]-[27] among which it is the Born-Infeld electrodynamics which is of interest for the present study.

In a recent paper [26], analytical investigations of some properties of s-wave holographic superconductors in a Schwarzschild-AdS background spacetime in the framework of Born-Infeld electrodynamics were carried out. The computation to obtain the relation between the critical temperature and the charge density and also the dependence of the condensation operator on the Born-Infeld parameter b (for temperatures $T < T_c$) was performed upto first order in b . The boundary condition imposed in the analysis was $\langle \mathcal{O} \rangle = \psi^-$ and $\langle \mathcal{O} \rangle = \psi^+ = 0$ which implies that the condensation operator \mathcal{O} in the boundary field theory corresponds to an operator with conformal dimension $\Delta_- = 1$. The analytical results were in good agreement with the numerical findings. However, there is another boundary condition, $\langle \mathcal{O} \rangle = \psi^+$ and $\langle \mathcal{O} \rangle = \psi^- = 0$ which implies that the condensation operator \mathcal{O} in the boundary

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field theory corresponds to an operator with conformal dimension $\Delta_+ = 2$. The analytical treatment involving this boundary condition has not been presented either in [26] or elsewhere. Indeed, such an analysis is worth carrying out since the analytical treatment in this case is far more involved than the one involving the other boundary condition.

To begin with, we consider a fixed planar Schwarzschild-AdS black hole background which reads¹

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(dx^2 + dy^2) \quad (1)$$

where the metric function is

$$f(r) = \left(r^2 - \frac{r_+^3}{r}\right) \quad (2)$$

r_+ being the horizon radius of the black hole. The Hawking temperature of the black hole may be written as

$$T = \frac{1}{4\pi} \left(\frac{\partial f(r)}{\partial r} \right)_{r=r_+} = \frac{3r_+}{4\pi} \quad (3)$$

In the presence of electric field and a complex scalar field ($\psi(r)$) we may write the corresponding lagrangian density as

$$\mathcal{L} = \mathcal{L}_{BI} - |\nabla_\mu \psi - iqA_\mu \psi|^2 - m^2 |\psi|^2 \quad (4)$$

where \mathcal{L}_{BI} is the Born-Infeld lagrangian density given by

$$\mathcal{L}_{BI} = \frac{1}{b} \left(1 - \sqrt{1 + \frac{bF}{2}} \right) \quad (5)$$

Here $F \equiv F^{\mu\nu} F_{\mu\nu}$ and b is the Born-Infeld parameter. The equation of motion for the electromagnetic field tensor $F_{\mu\nu}$ can be written as

$$\partial_\mu \left(\frac{\sqrt{-g} F^{\mu\nu}}{\sqrt{1 + \frac{bF}{2}}} \right) = \mathcal{J}^\nu. \quad (6)$$

Taking the ansatz[8] $\psi = \psi(r)$ and $A_\mu = (\phi(r), 0, 0, 0)$ we may write the equations of motion for the scalar field $\psi(r)$ and the electric scalar potential $\phi(r)$ as[22, 26]²

$$\psi''(r) + \left(\frac{f'}{f} + \frac{2}{r} \right) \psi'(r) + \left(\frac{\phi^2(r)}{f^2} + \frac{2}{f} \right) \psi(r) = 0 \quad (7)$$

$$\phi''(r) + \frac{2}{r} (1 - b\phi'^2(r)) \phi'(r) - \frac{2\psi^2(r)}{f} (1 - b\phi'^2(r))^{\frac{3}{2}} = 0. \quad (8)$$

Boundary conditions on $\phi(r)$ and $\psi(r)$ near the horizon $r \rightarrow r_+$ and at spatial infinity $r \rightarrow \infty$ needs to be imposed to solve the above equations. The regularity of ϕ and ψ at the horizon requires $\phi(r_+) = 0$ and $\psi(r_+) = -\frac{3r_+}{2}\psi'(r)[8]$.

Under the change of coordinates $z = \frac{r_+}{r}$ eq(s) (7) and (8) become

$$z\psi''(z) - \frac{2+z^3}{1-z^3}\psi'(z) + \left[z \frac{\phi^2(z)}{r_+^2(1-z^3)^2} + \frac{2}{z(1-z^3)} \right] \psi(z) = 0 \quad (9)$$

$$\phi''(z) + \frac{2bz^3}{r_+^2}\phi'(z) - \frac{2\psi^2(z)}{z^2(1-z^3)} \left(1 - \frac{bz^4}{r_+^2}\phi'^2(z) \right)^{\frac{3}{2}} \phi(z) = 0 \quad (10)$$

¹In this paper we have taken the AdS radius $l = 1$ and the gravitational constant $G = 1$.

²In our analysis we shall always take $m^2 = -2$.

The boundary condition $\phi(r_+) = 0$ now becomes $\phi(z = 1) = 0$. At the spatial infinity ϕ and ψ can be approximated as

$$\begin{aligned}\phi(z) &\approx \mu - \frac{\rho}{r} \\ &= \mu - \frac{\rho}{r_+} z\end{aligned}\tag{11}$$

and

$$\psi(z) \approx \frac{\psi^{(+)}}{r_{\Delta+}} + \frac{\psi^{(-)}}{r_{\Delta-}}\tag{12}$$

where $\Delta_{\pm} = \frac{3}{2} \pm \sqrt{\frac{9}{4} + m^2}$ is the conformal dimension of the *condensation operator* (\mathcal{O}) in the boundary field theory. Also $\psi^{(+)}$ and $\psi^{(-)}$ correspond to the vacuum expectation values of \mathcal{O} dual to the scalar field while μ and ρ are interpreted as the chemical potential and charge density of the dual field theory. In order to achieve stability in the asymptotic AdS region one sets any one of the ψ s to vanish. The case of vanishing of $\psi^{(+)}$ was studied earlier[26]. In the present work we shall focus on the condition $\psi^{(-)} = 0$. Hence $\langle \mathcal{O} \rangle = \psi^{(+)}$.

With the above set up in place, we now move on to investigate the relation between the critical temperature of condensation and the charge density.

At the critical temperature T_c the scalar field ψ vanishes, so eq.(10) becomes

$$\phi''(z) + \frac{2bz^3}{r_{+(c)}^2} \phi'(z) = 0.\tag{13}$$

The solution for this equation in the interval $[z, 1]$ reads[26]

$$\phi(z) = \lambda r_{+(c)} \xi(z)\tag{14}$$

where

$$\xi(z) = \int_z^1 \frac{d\tilde{z}}{\sqrt{1 + b\lambda^2 \tilde{z}^4}}.\tag{15}$$

We shall perform a perturbative expansion of $b\lambda^2$ in the r.h.s of eq. (15) and retain only the terms that are linear in b such that $b\lambda^2 = b\lambda_0^2 + O(b^2)$, where λ_0^2 is the value of λ^2 for $b = 0$. Now for our particular choice of $\psi^{(i)}$ ($i = +, -$) we have $\lambda_0^2 \approx 17.3$ [9]. Recalling that the existing values of b in the literature are $b = 0.1, 0.2, 0.3$ [22] we observe that $b\lambda_0^2 > 1$. Consequently the binomial expansion of the denominator in (15) has to be done carefully. The integration appearing in (15) is done for two ranges of values of z , one for $z \leq \Lambda < 1$ while the other for $\Lambda \leq z \leq 1$, where Λ is such that $b\lambda_0^2 z^4|_{z=\Lambda} = 1$. At this stage, it is to be noted that $b\lambda_0^2 z^4 < 1$ for $z < \Lambda$, whereas, on the other hand $b\lambda_0^2 z^4 > 1$ for $z > \Lambda$.

For the first case ($z \leq \Lambda < 1$),

$$\begin{aligned}\xi(z) = \xi_1(z) &= \int_z^{\Lambda} \frac{d\tilde{z}}{\sqrt{1 + b\lambda_0^2 \tilde{z}^4}} + \int_{\Lambda}^1 \frac{d\tilde{z}}{\sqrt{1 + b\lambda_0^2 \tilde{z}^4}} \\ &\approx \int_z^{\Lambda} \left(1 - \frac{b\lambda_0^2 \tilde{z}^4}{2}\right) + \frac{1}{\sqrt{b}\lambda_0} \int_{\Lambda}^1 \left(\frac{1}{\tilde{z}^2} - \frac{1}{2b\lambda_0^2 \tilde{z}^6}\right) \\ &\approx \left[\frac{9}{5}\Lambda - z + \frac{z^5}{10\Lambda^4} - \Lambda^2 + \frac{\Lambda^6}{10}\right].\end{aligned}\tag{16}$$

Similarly in the range $\Lambda \leq z \leq 1$, we have

$$\begin{aligned}\xi(z) = \xi_2(z) &= \int_z^1 \frac{d\tilde{z}}{\sqrt{1 + b\lambda_0^2 \tilde{z}^4}} \\ &\approx \frac{1}{\sqrt{b}\lambda_0} \int_z^1 \left(\frac{1}{\tilde{z}^2} - \frac{1}{2b\lambda_0^2 \tilde{z}^6} \right) \\ &\approx \frac{\Lambda^2}{z^5} \left[z^4(1-z) + \frac{\Lambda^4}{10}(z^5-1) \right].\end{aligned}\quad (17)$$

The consistency of our expressions may be seen by noting that at the common point $z = \Lambda$, both integrals (16) and (17) yield the same answer,

$$\xi_1(\Lambda) = \xi_2(\Lambda) = \frac{9}{10}\Lambda - \Lambda^2 + \frac{\Lambda^6}{10} \quad (18)$$

Furthermore, the boundary condition $\phi(1) = 0$ is also satisfied ($\xi_2(1) = 0$).

We may now express $\psi(z)$ near the boundary as

$$\psi(z) = \frac{<\mathcal{O}>}{\sqrt{2r_+^2}} z^2 \mathcal{F}(z) \quad (19)$$

with the condition $\mathcal{F}(0) = 1$ and $\mathcal{F}'(0) = 0$.

Using the above equation, eq.(9) may be written as

$$\mathcal{F}''(z) - \frac{(5z^4 - 2z)}{z^2(1 - z^3)} \mathcal{F}'(z) - \frac{4z^3}{z^2(1 - z^3)} \mathcal{F}(z) + \lambda^2 \frac{\xi^2(z)}{(1 - z^3)^2} \mathcal{F}(z) = 0. \quad (20)$$

This equation can be put in the Sturm-Liouville form

$$[p(z)\mathcal{F}'(z)]' + q(z)\mathcal{F}(z) + \lambda^2 g(z)\mathcal{F}(z) = 0 \quad (21)$$

with the identifications

$$\begin{aligned}p(z) &= z^2(1 - z^3) \\ q(z) &= -4z^3 \\ g(z) &= \frac{z^2}{(1 - z^3)} \xi^2(z) = \chi(z)\xi^2(z)\end{aligned} \quad (22)$$

where $\chi(z) = \frac{z^2}{(1 - z^3)}$.

Using eq.(22), we may write the eigenvalue λ^2 as

$$\begin{aligned}\lambda^2 &= \frac{\int_0^1 \{p(z)[\mathcal{F}'(z)]^2 - q(z)[\mathcal{F}(z)]^2\} dz}{\int_0^1 \{g(z)[\mathcal{F}(z)]^2\} dz} \\ &= \frac{\int_0^1 \{p(z)[\mathcal{F}'(z)]^2 - q(z)[\mathcal{F}(z)]^2\} dz}{\int_0^\Lambda \{\chi(z)\xi_1^2(z)[\mathcal{F}(z)]^2\} dz + \int_\Lambda^1 \{\chi(z)\xi_2^2(z)[\mathcal{F}(z)]^2\} dz}.\end{aligned} \quad (23)$$

Choosing the trial function

$$\mathcal{F}(z) = 1 - \alpha z^2 \quad (24)$$

which satisfies the conditions $\mathcal{F}(0) = 1$ and $\mathcal{F}'(0) = 0$, we shall now determine the eigenvalues for different values of the parameter b .

For $b = 0.1$, we obtain

$$\lambda^2 = 300.769 + \frac{2.27395\alpha - 5.19713}{0.0206043 + (0.00265985\alpha - 0.0119935)\alpha} \quad (25)$$

which has a minima for $\alpha \approx 0.653219$. Therefore from eq.(23) we obtain

$$\lambda^2 \approx 33.8298 \quad (26)$$

Thus the critical temperature for condensation (T_c) in terms of the charge density (ρ) can be obtained as

$$T_c = \frac{3r_{+(c)}}{4\pi} = \frac{3}{4\pi} \sqrt{\frac{\rho}{\lambda}} \approx 0.099\sqrt{\rho} \quad (27)$$

This value of T_c is indeed in very good agreement with the exact value $T_c = 0.10072\sqrt{\rho}$ [22]. Similarly, for the other values of the Born-Infeld parameter (b), we obtain the corresponding analytic values for the coefficients of T_c which are presented in the table below.

Values of b	Numerical values of T_c	Analytic values of T_c
0.1	$0.10072\sqrt{\rho}$	$0.099\sqrt{\rho}$
0.2	$0.08566\sqrt{\rho}$	$0.093\sqrt{\rho}$
0.3	$0.07292\sqrt{\rho}$	$0.089\sqrt{\rho}$

In this letter, based on Sturm-Liouville (SL) eigenvalue method we have performed analytic computations on s -wave holographic superconductor in the probe limit, where the usual Maxwell action is replaced by the Born-Infeld (BI) action. Based on this analytic method we have been to establish the relationship between the critical temperature (T_c) and the charge density (ρ) for the corresponding boundary field theory. The analytic results thus obtained are found to be in good agreement to that with the existing numerical results[22]. As a future project, it would be quite interesting to see whether the present analysis could be extended to study the behavior of s -wave order parameter near the critical point.

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